

CARLEMAN ESTIMATES AND NULL CONTROLLABILITY OF COUPLED DEGENERATE SYSTEMS

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ABSTRACT. In this paper, we study the null controllability of weakly degenerate coupled parabolic systems with two different diffusion coefficients and one control force. To obtain this aim, we develop first new global Carleman estimates for degenerate parabolic equations with weight functions different from the ones of [2], [10] and [32].

1. Introduction

This paper is concerned with the null controllability for the coupled degenerate parabolic systems

$$u_t - (x^{\alpha_1} u_x)_x + b_{11}(t, x)u + b_{12}(t, x)v = h(t, x)1_\omega, \quad (t, x) \in \times(0, 1), \quad (1.1)$$

$$v_t - (x^{\alpha_2} v_x)_x + b_{22}(t, x)v + b_{21}(t, x)u = 0, \quad (t, x) \in \times(0, 1), \quad (1.2)$$

$$u(t, 0) = u(t, 1) = v(t, 0) = v(t, 1) = 0, \quad t \in (0, T), \quad (1.3)$$

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in (0, 1), \quad (1.4)$$

where $\omega = (a, b)$ is an open subset of $(0, 1)$, $h \in L^2((0, T) \times [0, 1))$, $(u_0, v_0) \in L^2(0, 1) \times L^2(0, 1)$, $(\alpha_1, \alpha_2) \in (0, 1)^2$ and $b_{ij} \in L^\infty((0, T) \times (0, 1))$, $i, j = 1, 2$.

Controllability properties of nondegenerate parabolic equations have been widely studied, see [6], [15], [18], [19], [20], [21], [22], [28], [30], [31], [33], [34], using several techniques in particular the Carleman estimates. In [2], [10], [32] new Carleman estimates were developed for degenerate parabolic equations and used to show observability inequalities of the adjoint degenerate problems and then obtain the null controllability. Recently, in [14] Cannarsa et al. established a local Carleman estimate and deduced unique continuation and boundary approximate controllability for weakly degenerate equations.

The null controllability of coupled parabolic systems was studied for example in [4], [5], [24], [25], [26], [27] in the nondegenerate case. In [29], Liu et al. considered parabolic cascade systems, $b_{12} = 0$, with degeneracy in only one equation, using the nondegenerate Carleman estimate of Fursikov and Imanuvilov [23] and an approximation argument as in [13]. In [8], Cannarsa and De Teresa studied the null controllability of cascade degenerate linear systems with the same diffusion coefficient, i.e., $\alpha_1 = \alpha_2$, and with the particular coupling term $b_{21} = 1_O$ for some open set $O \subset (0, 1)$. In [1], we studied the null controllability for degenerate cascade systems with general coupling terms and two different diffusion coefficients. We used a Carleman estimate from [2], and chose carefully appropriate parameters in the weight functions $\varphi_1(t, x) = \frac{\lambda_1(x^{2-\alpha_1}-d_1)}{t^4(T-t)^4}$ and $\varphi_2(t, x) = \frac{\lambda_2(x^{2-\alpha_2}-d_2)}{t^4(T-t)^4}$ to obtain the inequality $e^{s\varphi_1} \leq Ce^{s\varphi_2}$ to absorb the coupling term.

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For general degenerate systems (1.1)-(1.4), we need the uniform equivalence $e^{s\varphi^1} \equiv e^{s\varphi^2}$. But this occurs if and only if $\alpha_1 = \alpha_2$. To overcome this problem we propose in this paper a common weight function $\varphi(t, x) = \frac{\lambda(x^{2-\beta}-d)}{t^k(T-t)^k}$ for some β in terms of α_1 and α_2 . Then, the first step in this paper is to show new Carleman estimates for the following degenerate parabolic equation

$$y_t - (x^\alpha y_x)_x = f(t, x), \quad (t, x) \in (0, T) \times (0, 1), \quad (1.5)$$

$$y(t, 0) = y(t, 1) = 0, \quad t \in (0, T), \quad (1.6)$$

$$y(0, x) = y_0, \quad x \in (0, 1), \quad (1.7)$$

with the weight function $\varphi(t, x) = \frac{\lambda(x^{2-\beta}-d)}{t^k(T-t)^k}$ with d, λ and k constants to be specified later. To prove our Carleman estimates, we need to show the following fundamental Hardy-Poincaré inequality

$$\int_0^1 x^{\gamma-2} v^2 dx \leq C_\gamma \int_0^1 x^\gamma v_x^2 dx \quad \text{where } C_\gamma = \frac{4}{(1-\gamma)^2} \quad (1.8)$$

for $\gamma < 1$, and v satisfying $v(0) = 0$ and $\int_0^1 x^\gamma v_x^2 dx < +\infty$. This result was proved in [2], [10] and [32] for $0 < \gamma < 2, \gamma \neq 1$. But, for our Carleman estimates we need this inequality for negative γ , see Lemma 6.1. This will allow us to deduce Carleman estimates for the adjoint coupled degenerate system

$$U_t - (x^{\alpha_1} U_x)_x + b_{11}(t, x)U + b_{21}(t, x)V = 0, \quad (t, x) \in (0, T) \times (0, 1), \quad (1.9)$$

$$V_t - (x^{\alpha_2} V_x)_x + b_{22}(t, x)V + b_{12}(t, x)U = 0, \quad (t, x) \in (0, T) \times (0, 1), \quad (1.10)$$

$$U(t, 1) = U(t, 0) = V(t, 1) = V(t, 0) = 0, \quad t \in (0, T), \quad (1.11)$$

$$U(0, x) = U_0(x), V(0, x) = V_0(x), \quad x \in (0, 1), \quad (1.12)$$

and then its observability inequality. Using a standard argument, we obtain the null controllability of (1.1)-(1.4). By a linearization argument and fixed point, see for example [1], [2], [9], [35] one can show easily the null controllability of semilinear degenerate coupled systems.

This paper is organized as follows. Section 2 is devoted to the well-posedness of the coupled degenerate systems. In section 3, we establish our new Carleman estimates for degenerate parabolic equations and deduce similar estimates for the coupled degenerate systems. In section 4, we deduce observability inequality and null controllability results. In appendix, we give summarized proofs of Caccioppoli and Hardy-Poincaré inequalities.

2. Well-posedness

In order to study the well-posedness of the system (1.1)-(1.4), we introduce the weighted spaces

$$H_{\alpha_i}^1(0, 1) := \left\{ u \in L^2(0, 1) : u \text{ is abs. continuous in } [0, 1], x^{\alpha_i/2} u_x \in L^2(0, 1) \text{ and } u(0) = u(1) = 0 \right\}$$

with the norm $\|u\|_{H_{\alpha_i}^1(0, 1)}^2 := \|u\|_{L^2(0, 1)}^2 + \|x^{\alpha_i/2} u_x\|_{L^2(0, 1)}^2$ and

$$H_{\alpha_i}^2(0, 1) := \left\{ u \in H_{\alpha_i}^1(0, 1) : x^{\alpha_i} u_x \in H^1(0, 1) \right\}$$

with the norm

$$\|u\|_{H_{\alpha_i}^2(0, 1)}^2 := \|u\|_{H_{\alpha_i}^1(0, 1)}^2 + \|(x^{\alpha_i} u_x)_x\|_{L^2(0, 1)}^2.$$

We define the operator $(A_i, D(A_i))$ by

$$A_i u := (x^{\alpha_i} u_x)_x, \quad u \in D(A_i) = H_{\alpha_i}^2(0, 1), \quad i = 1, 2.$$

We recall the following properties of $(A_i, D(A_i))$.

Proposition 2.1. ([7], [13]). *For $i = 1, 2$, the operator $A_i : D(A_i) \rightarrow L^2(0, 1)$ is closed, self-adjoint, negative and with dense domain.*

In the Hilbert space $\mathbb{H} := L^2(0, 1) \times L^2(0, 1)$, the system (1.1)-(1.4) can be transformed in the following Cauchy problem

$$(CP) \quad \begin{cases} X'(t) = \mathcal{A}X(t) + B(t)X(t) + G(t), \\ X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \end{cases}$$

where $X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $D(\mathcal{A}) = D(A_1) \times D(A_2)$, $G(t) = \begin{pmatrix} h(t, x)^{1\omega} \\ 0 \end{pmatrix}$, and

$$B(t) = \begin{pmatrix} M_{b_{11}(t)} & M_{b_{12}(t)} \\ M_{b_{21}(t)} & M_{b_{22}(t)} \end{pmatrix}, \text{ where } M_{b_{ij}(t)}u = b_{ij}(t)u.$$

As the operator \mathcal{A} is diagonal and since $B(t)$ is a bounded perturbation, the following wellposedness and regularity results hold.

Proposition 2.2. (i) *The operator \mathcal{A} generates a contraction strongly continuous semigroup $(T(t))_{t \geq 0}$.*
(ii) *For all $h \in L^2((0, T) \times (0, 1))$ and $(u_0, v_0) \in L^2(0, 1) \times L^2(0, 1)$ there exists a unique mild solution $(u, v) \in X_T := C([0, T], L^2(0, 1) \times L^2(0, 1)) \cap L^2(0, T; H_{\alpha_1}^1 \times H_{\alpha_2}^1)$ of (1.1)-(1.4) satisfying*

$$\begin{aligned} & \sup_{[0, T]} \|(u, v)(t)\|_{L^2 \times L^2}^2 + \int_0^T \|(x^{\frac{\alpha_1}{2}} u_x, x^{\frac{\alpha_2}{2}} v_x)\|_{L^2}^2 dt \\ & \leq C_T \left(\|(u_0, v_0)\|_{L^2 \times L^2}^2 + \|h\|_{L^2((0, T) \times (0, 1))}^2 \right) \end{aligned} \quad (2.13)$$

for a constant $C_T > 0$. Moreover, if $(u_0, v_0) \in H_{\alpha_1}^1 \times H_{\alpha_2}^1$ then, $(u, v) \in Y_T := C([0, T], H_{\alpha_1}^1 \times H_{\alpha_2}^1) \cap H^1(0, T; L^2(0, 1) \times L^2(0, 1)) \cap L^2(0, T; H_{\alpha_1}^2 \times H_{\alpha_2}^2)$ and

$$\begin{aligned} & \sup_{[0, T]} \|(u, v)(t)\|_{H_{\alpha_1}^1 \times H_{\alpha_2}^1}^2 + \int_0^T (\|(u_t, v_t)\|_{L^2}^2 + \|((x^{\alpha_1} u_x)_x, (x^{\alpha_2} v_x)_x)\|_{L^2}^2) dt \\ & \leq C_T \left(\|(u_0, v_0)\|_{H_{\alpha_1}^1 \times H_{\alpha_2}^1}^2 + \|h\|_{L^2((0, T) \times (0, 1))}^2 \right) \end{aligned}$$

for a constant $C_T > 0$.

3. Carleman estimates

In this section we prove new Carleman estimates for the adjoint system (1.9)-(1.12). For this, let $\omega' := (a', b') \Subset \omega$ and let us introduce the weight functions : $\varphi(t, x) := \Theta(t)\psi(x)$; $\Theta(t) := \frac{1}{t^k(T-t)^k}$; $\psi(x) := \lambda(x^{2-\beta} - d)$; $\Phi(t, x) = \Psi(x)\Theta(t)$; $\Psi(x) := (e^{\rho\sigma(x)} - e^{2\rho\|\sigma\|_\infty})$; $\phi(t, x) = e^{\rho\sigma(x)}\Theta(t)$; where σ is a function in $C^2([a', 1])$ satisfying $\sigma(x) > 0$ in $(a', 1)$, $\sigma(a') = \sigma(1) = 0$ and $\sigma_x(x) \neq 0$ in $[a', 1] \setminus \omega_0$ for some open $\omega_0 \Subset (a', 1)$ and the parameters d, ρ, λ and k are chosen such that $d \geq 5$; $\rho > \frac{4\ln 2}{\|\sigma\|_\infty}$, $\frac{e^{2\rho\|\sigma\|_\infty}}{d-1} < \lambda < \frac{4}{3d}(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty})$ and $k \geq 4$.

Remark 3.1. • *These weight functions are independent of the diffusion coefficient. This play a crucial role to study coupled system of non cascade form.*

• *The existence of the function σ was proved for example in [23] using Morse functions. But in 1-dimension one can show this easily using cut-off functions.*

- If $d \geq 5$ and $\rho > \frac{4\ln 2}{\|\sigma\|_\infty}$ then the interval $\left] \frac{e^{2\rho\|\sigma\|_\infty}}{d-1}, \frac{4}{3d}(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}) \right[$ is not empty. We can then choose λ in this interval.
- For this choice of the parameters d, ρ and λ the weight functions φ and Φ satisfy the following inequalities which are needed in the sequel

$$\frac{4}{3}\Phi < \varphi < \Phi \text{ on } (0, T) \times (0, 1). \quad (3.14)$$

- For nondegenerate problems one needs the following estimates see e.g. [23]

$$\lim_{t \rightarrow 0^+} \Theta(t) = \lim_{t \rightarrow T^-} \Theta(t) = +\infty, \quad \Theta(t) \geq c_1, \quad |\dot{\Theta}| \leq c_2 \Theta^2, \quad |\ddot{\Theta}| \leq c_3 \Theta^3. \quad (3.15)$$

and this is satisfied for all $k \geq 1$ with $c_1 = (2/T)^{2k}$, $c_2 = kT(T/2)^{2(k-1)}$, $c_3 = k(k+1)T^2(T/2)^{4(k-1)}$.

- For the degenerate case one needs in addition the estimate

$$|\ddot{\Theta}| \leq c_4 \Theta^2. \quad (3.16)$$

which is satisfied for all $k \geq 2$ with $c_4 = k(k+1)T^2(T/2)^{k-4}$.

We begin by proving first a new Carleman estimate for the problem (1.5)-(1.7) with one equation.

Theorem 3.2. *Let $T > 0$ and suppose that $y_0 \in H_\alpha^1$. Then, for all $\beta \in [\alpha, 1)$ there exist two positive constants C and s_0 such that every solution y of (1.5)-(1.7) satisfies for all $s \geq s_0$*

$$\begin{aligned} & \int_0^T \int_0^1 \left(s\Theta(t)x^{2\alpha-\beta}y_x^2 + s^3\Theta^3(t)x^{2+2\alpha-3\beta}y^2 \right) e^{2s\varphi(t,x)} dx dt \\ & \leq C \left(\int_0^T \int_0^1 f^2(t,x)e^{2s\varphi(t,x)} dx dt + \int_0^T s\Theta(t)y_x^2(t,1)e^{2s\varphi(t,1)} dt \right). \end{aligned} \quad (3.17)$$

Proof. For $s > 0$, let us introduce the function $z := e^{s\varphi}y$. We have

$$L_s z := z_t + (x^\alpha z_x)_x - 2sx^\alpha \varphi_x z_x - s\varphi_t z + s^2 x^\alpha \varphi_x^2 z - s(x^\alpha \varphi_x)_x z = f e^{s\varphi}.$$

Let

$$\begin{aligned} L_s^+ z &:= (x^\alpha z_x)_x - s\varphi_t z + s^2 x^\alpha \varphi_x^2 z, \\ L_s^- z &:= z_t - 2sx^\alpha \varphi_x z_x - s(x^\alpha \varphi_x)_x z, \\ f_s &:= f e^{s\varphi}. \end{aligned}$$

We have $\|f_s\|_{L^2}^2 = \|L_s^+ z + L_s^- z\|_{L^2}^2 = \|L_s^+ z\|_{L^2}^2 + \|L_s^- z\|_{L^2}^2 + 2\langle L_s^+ z, L_s^- z \rangle \geq 2\langle L_s^+ z, L_s^- z \rangle$. One has $z(0, x) = z(T, x) = z_x(0, x) = z_x(T, x) = 0$. So integrating by parts one obtains

$$\begin{aligned} \langle L_s^+ z, L_s^- z \rangle &= -2s^2 \int_0^T \int_0^1 x^\alpha \varphi_x \varphi_{tx} z^2 dx dt + s \int_0^T \int_0^1 x^\alpha (x^\alpha \varphi_x)_{xx} z z_x dx dt \\ &= \frac{s}{2} \int_0^T \int_0^1 \varphi_{tt} z^2 dx dt + s \int_0^T \int_0^1 x^\alpha [2x^\alpha \varphi_{xx} + \alpha x^{\alpha-1} \varphi_x] z_x^2 dx dt \\ &\quad + s^3 \int_0^T \int_0^1 x^\alpha [2x^\alpha \varphi_{xx} + \alpha x^{\alpha-1} \varphi_x] \varphi_x^2 z^2 dx dt \\ &\quad + \int_0^T \left[x^\alpha z_x z_t + s^2 \Theta \dot{\Theta} \psi_x x^\alpha z^2 - s^3 \Theta^3 x^{2\alpha} \psi_x^3 z^2 \right]_{x=0}^{x=1} dt \\ &\quad - \int_0^T \left[\lambda(2-\beta)s\Theta x^{1-\beta} (x^\alpha z_x)^2 + \lambda(2-\beta)(1+\alpha-\beta)s\Theta x^{2\alpha-\beta} z z_x \right]_{x=0}^{x=1} dt. \end{aligned}$$

It is easy to check that if $y \in H_\alpha^2(0, 1)$ then we have also $z \in H_\alpha^2(0, 1)$. So $x^\alpha z \in H^1(0, 1) \subset L^\infty(0, 1)$ by the Sobolev imbedding theorem. Then, using the facts that $z(t, 0) = z(t, 1) = z_t(t, 0) = z_t(t, 1) = 0$ and $x^\alpha z_x$, x^α , ψ , ψ_x are bounded, we deduce that the first integral with boundary terms vanishes and $x^{1-\beta}(x^\alpha z_x)^2|_{x=0} = 0$. On the other hand we have $[x^{2\alpha-\beta} z z_x]_{x=0}^{x=1} = 0$, in fact it is clear that $x^{2\alpha-\beta} z z_x|_{x=1} = (x^\alpha z_x) z|_{x=1} = 0$ and since $x^\alpha z_x \in L^\infty(0, 1)$ and $z(t, 0) = 0$ then for each $t \in (0, T)$ we have

$$|z_x(t, x)| \leq cx^{-\alpha} \text{ and } |z(t, x)| = \left| \int_0^x z_x(t, y) dy \right| \leq cx^{1-\alpha}. \quad (3.18)$$

Therefore $|x^{2\alpha-\beta} z z_x(t, x)| \leq cx^{1-\beta}$. Consequently, since $\beta < 1$ we deduce $x^{2\alpha-\beta} z z_x|_{x=0} = 0$.

We have then

$$\begin{aligned} & \underbrace{\lambda^3(2-\beta)^3(2-2\beta+\alpha) \int_0^T \int_0^1 s^3 \Theta^3 x^{2+2\alpha-3\beta} z^2 dx dt}_{J_1} + \underbrace{\lambda(2-\beta)(2-2\beta+\alpha) \int_0^T \int_0^1 s \Theta x^{2\alpha-\beta} z_x^2 dx dt}_{J_2} \\ & \leq \frac{1}{2} \int_0^T \int_0^1 f^2 e^{2s\varphi} dx dt - \underbrace{2\lambda^2(2-\beta)^2 \int_0^T \int_0^1 s^2 \Theta \dot{\Theta} x^{2+\alpha-2\beta} z^2 dx dt}_{J_3} \\ & + \underbrace{\lambda(2-\beta)(1+\alpha-\beta)(\beta-\alpha) \int_0^T \int_0^1 s \Theta x^{2\alpha-\beta-1} z z_x dx dt}_{J_4} + \underbrace{\frac{\lambda}{2} \int_0^T \int_0^1 s \ddot{\Theta} (d-x^{2-\beta}) z^2 dx dt}_{J_5} \\ & + \lambda(2-\beta) \int_0^T s \Theta z_x^2(t, 1) dt. \end{aligned}$$

Now we will show that J_3 , J_4 and J_5 can be absorbed by J_1 and J_2 . For this, let $\varepsilon > 0$ fixed to be specified later. First, Since $\beta \geq \alpha$ and $|\Theta \dot{\Theta}| \leq C \Theta^3$ then

$$|J_3| \leq C \int_0^T \int_0^1 s^2 \Theta^3 x^{2+2\alpha-3\beta} z^2 dx dt \leq \varepsilon J_1$$

for s large enough. In the other hand for J_4 we have

$$\begin{aligned} |J_4| & \leq \lambda(2-\beta)(1+\alpha-\beta)(\beta-\alpha) \int_0^T \int_0^1 \left[\sqrt{s\Theta} x^{\alpha-\frac{\beta}{2}-1} |z| \right] \left[\sqrt{s\Theta} x^{\alpha-\frac{\beta}{2}} |z_x| \right] dx dt \\ & \leq \lambda(2-\beta)(1+\alpha-\beta)(\beta-\alpha) \left(\varepsilon \int_0^T \int_0^1 s \Theta x^{2\alpha-\beta-2} z^2 dx dt + \frac{1}{4\varepsilon} \int_0^T \int_0^1 s \Theta x^{2\alpha-\beta} z_x^2 dx dt \right) \end{aligned} \quad (3.19)$$

Now we will use the Hardy-Poincaré inequality (6.56). We have $2\alpha - \beta < 1$ and we will show that $\int_0^1 x^{2\alpha-\beta} z_x^2 dx < +\infty$. Using (3.18) and the fact that $\beta < 1$ we obtain,

$$|x^{2\alpha-\beta} z_x^2| \leq C x^{-\beta} \in L^1(0, 1).$$

We have then

$$\int_0^1 x^{2\alpha-\beta-2} z^2 dx \leq C_{2\alpha-\beta} \int_0^1 x^{2\alpha-\beta} z_x^2 dx$$

where $C_{2\alpha-\beta} = \frac{4}{(1-2\alpha+\beta)^2}$. Then, we get from (3.19)

$$|J_4| \leq \lambda(2-\beta)(1+\alpha-\beta)(\beta-\alpha) \left(\varepsilon C_{2\alpha-\beta} + \frac{1}{4\varepsilon} \right) \int_0^T \int_0^1 s \Theta x^{2\alpha-\beta} z_x^2 dx dt$$

The quantity $\varepsilon C_{2\alpha-\beta} + \frac{1}{4\varepsilon}$ is minimal for $\varepsilon = \frac{1}{2\sqrt{C_{2\alpha-\beta}}}$. For this choice we have

$$|J_4| \leq \lambda(2-\beta)(1+\alpha-\beta)(\beta-\alpha) \frac{2}{1-2\alpha+\beta} \int_0^T \int_0^1 s\Theta x^{2\alpha-\beta} z_x^2 dx dt$$

and for all $\beta \in [\alpha, 1)$ we have

$$\frac{2(1+\alpha-\beta)(\beta-\alpha)}{1-2\alpha+\beta} - (2-2\beta+\alpha) = \frac{(\beta-1)(2-\alpha)}{1-2\alpha+\beta} < 0$$

The term J_4 can then be absorbed by J_2 .

For the last term J_5 , since $|\dot{\Theta}| \leq c_4\Theta^2$ and $\beta \geq \alpha$, we have by applying the Hardy-Poincaré inequality

$$\begin{aligned} |J_5| &\leq \lambda dc_4 \int_0^T \int_0^1 s\Theta^2 z^2 dx dt \\ &= \lambda dc_4 \int_0^T \int_0^1 \left[\sqrt{s\Theta} x^{\alpha-\frac{\beta}{2}-1} z \right] \left[\sqrt{s\Theta}^{\frac{3}{2}} x^{1-\alpha+\frac{\beta}{2}} z \right] dx dt \\ &\leq \lambda dc_4 \int_0^T \int_0^1 \left(\varepsilon s\Theta x^{2\alpha-\beta-2} z^2 + \frac{1}{4\varepsilon} s\Theta^3 x^{2-2\alpha+\beta} z^2 \right) dx dt \\ &\leq \varepsilon \lambda dc_4 C_{2\alpha-\beta} \int_0^T \int_0^1 s\Theta x^{2\alpha-\beta} z_x^2 dx dt + C_\varepsilon \int_0^T \int_0^1 s\Theta^3 x^{2+2\alpha-3\beta} z^2 dx dt \end{aligned}$$

Therefore by choosing ε small enough, we obtain

$$\begin{aligned} &\int_0^T \int_0^1 s^3 \Theta^3 x^{2+2\alpha-3\beta} z^2 dx dt + \int_0^T \int_0^1 s\Theta x^{2\alpha-\beta} z_x^2 dx dt \\ &\leq C \left(\int_0^T \int_0^1 f^2 e^{2s\varphi} dx dt + \int_0^T s\Theta z_x^2(t, 1) dt \right). \end{aligned}$$

for s large enough. So replacing z by $e^{s\varphi}y$ we deduce immediately the conclusion of the theorem. \square

Theorem 3.3. *Let $T > 0$ and suppose that $y_0 \in H_\alpha^1$. Then, for all $\beta \in [\alpha, 1)$ there exist two positive constants C and s_0 such that every solution y of (1.5)-(1.7) satisfies for all $s \geq s_0$*

$$\begin{aligned} &\int_0^T \int_0^1 \left(s\Theta(t) x^{2\alpha-\beta} y_x^2 + s^3 \Theta^3(t) x^{2+2\alpha-3\beta} y^2 \right) e^{2s\varphi(t,x)} dx dt \\ &\leq C \left(\int_0^T \int_0^1 f^2(t, x) e^{2s\Phi(t,x)} dx dt + \int_0^T \int_{\omega'} s^3 \phi^3 y^2 e^{2s\Phi(t,x)} dx dt \right) \end{aligned} \quad (3.20)$$

Proof. Let us consider an arbitrary open subset $\omega'' := (a'', b'') \Subset \omega'$ and a cut-off function $\xi \in \mathcal{C}^\infty(0, 1)$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & x \in (0, 1), \\ \xi(x) = 1, & 0 \leq x \leq a'', \\ \xi(x) = 0, & b'' \leq x \leq 1. \end{cases}$$

Let $z = \xi y$ where y is the solution of (1.5)-(1.7). Then z satisfies the following system

$$z_t - (x^\alpha z_x)_x = \xi f - \xi_x x^\alpha y_x - (x^\alpha \xi_x y)_x, \quad (t, x) \in (0, T) \times (0, 1), \quad (3.21)$$

$$z(t, 1) = z(t, 0) = 0, \quad t \in (0, T), \quad (3.22)$$

Therefore, applying the Carleman estimate (3.17) to the equation (3.21) we obtain

$$\begin{aligned} & \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha-\beta}z_x^2(t,x) + s^3\Theta^3(t)x^{2+2\alpha-3\beta}z^2(t,x)]e^{2s\varphi} dx dt \\ & \leq C \int_0^T \int_0^1 [\xi^2 f^2 + (\xi_x x^\alpha y_x + (x^\alpha \xi_x y)_x)^2] e^{2s\varphi} dx dt. \end{aligned} \quad (3.23)$$

So using the definition of ξ and the Cacciopoli's inequality, see Lemma 5.1, we obtain

$$\begin{aligned} \int_0^T \int_0^1 (\xi_x x^\alpha y_x + (x^\alpha \xi_x y)_x)^2 e^{2s\varphi} dx dt & \leq C \int_0^T \int_{\omega'} [y^2 + y_x^2] e^{2s\varphi} dx dt \\ & \leq C \int_0^T \int_{\omega'} y^2 e^{2s\varphi} dx dt. \end{aligned} \quad (3.24)$$

and

$$\int_0^T \int_0^1 s\Theta x^{2\alpha-\beta} \xi^2 y_x^2 e^{2s\varphi} dx dt \leq 2 \int_0^T \int_0^1 s\Theta x^{2\alpha-\beta} z_x^2 e^{2s\varphi} dx dt + 2 \int_0^T \int_{\omega'} s\Theta y^2 e^{2s\varphi} dx dt \quad (3.25)$$

Thus from (3.23)-(3.24) and the definition of ξ we deduce the following estimate

$$\begin{aligned} & \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha-\beta}\xi^2 y_x^2(t,x) + s^3\Theta^3(t)x^{2+2\alpha-3\beta}\xi^2 y^2(t,x)]e^{2s\varphi} dx dt \\ & \leq C \left(\int_0^T \int_0^1 \xi^2 f^2 e^{2s\varphi} dx dt + \int_0^T \int_{\omega'} s\Theta y^2 e^{2s\varphi} dx dt \right). \end{aligned} \quad (3.26)$$

On $(a', 1)$ the equation (1.5) is uniformly parabolic hence, one can use the following Carleman estimate which is a consequence of ([23], Lemma 1.2) established by Fursikov and Imanuvilov.

Proposition 3.4. *Consider the nondegenerate linear problem*

$$\begin{cases} v_t - (x^\alpha v_x)_x = f \in L^2((0, T) \times (a', 1)), \\ v(t, a') = v(t, 1) = 0, \quad t \in (0, T), \end{cases}$$

Then, there exists a constant $\rho_0 > 0$ such that for all $\rho \geq \rho_0$ there exists $s_0(\rho) > 0$ such that for each $s \geq s_0(\rho)$ the solution v of the last problem satisfy the following estimate:

$$\begin{aligned} & \int_0^T \int_{a'}^1 (s\phi v_x^2 + s^3\phi^3 v^2) e^{2s\Phi} dx dt \\ & \leq C \left(\int_0^T \int_{a'}^1 f^2 e^{2s\Phi} dx dt + \int_0^T \int_{\omega'} s^3\phi^3 v^2 e^{2s\Phi} dx dt \right) \end{aligned} \quad (3.27)$$

where the functions Φ and ϕ are defined in Theorem 3.3.

Remark 3.5. The last estimate was showed in [23] for $\Theta(t) = \frac{1}{t(T-t)}$ but by careful examination of the proof one can see easily that it remains valid for all $\Theta \in C^2(0, T)$ satisfying (3.15), see Remark 3.1.

To achieve the proof of the Theorem 3.7, let $Z := \zeta y$, where the function ζ is defined as $\zeta = 1 - \xi$. Then Z is a solution of the following problem

$$\begin{aligned} Z_t - (x^\alpha Z_x)_x &= \zeta f - \zeta_x x^\alpha y_x - (x^\alpha \zeta_x y)_x, & (t, x) &\in (0, T) \times (a', 1), \\ Z(t, 1) = Z(t, a') &= 0, & t &\in (0, T), \end{aligned}$$

Applying the classical Carleman estimate (3.27), it follows that for s large enough

$$\begin{aligned} & \int_0^T \int_0^1 (s\phi Z_x^2 + s^3\phi^3 Z^2) e^{2s\Phi} dx dt \\ & \leq C \left(\int_0^T \int_0^1 [\zeta f + \zeta_x x^\alpha y_x + (x^\alpha \zeta_x y)_x]^2 e^{2s\Phi} dx dt + \int_0^T \int_{\omega'} s^3\phi^3 Z^2 e^{2s\Phi} dx dt \right) \\ & \leq C \left(\int_0^T \int_0^1 \zeta^2 f^2 e^{2s\Phi} dx dt + \int_0^T \int_{\omega''} [y^2 + y_x^2] e^{2s\Phi} dx dt + \int_0^T \int_{\omega'} s^3\phi^3 Z^2 e^{2s\Phi} dx dt \right) \end{aligned}$$

Therefore, using the Caccioppoli inequality and the definitions of Z and ζ we deduce

$$\begin{aligned} & \int_0^T \int_0^1 (s\phi \zeta^2 y_x^2 + s^3\phi^3 \zeta^2 y^2) e^{2s\Phi} dx dt \\ & \leq C \left(\int_0^T \int_0^1 \zeta^2 f^2 e^{2s\Phi} dx dt + \int_0^T \int_{\omega'} s^3\phi^3 y^2 e^{2s\Phi} dx dt \right) \end{aligned} \quad (3.28)$$

Thanks to (3.14) there exists a constant $c > 0$ such that for all $(t, x) \in [0, T] \times (a', 1)$ one has

$$\Theta x^{2\alpha-\beta} e^{2s\varphi(t,x)} \leq c\phi e^{2s\Phi(t,x)} \text{ and } \Theta^3 x^{2+2\alpha-3\beta} e^{2s\varphi(t,x)} \leq c\phi^3 e^{2s\Phi(t,x)} \quad (3.29)$$

Then, using (3.26), (3.28), (3.14), (3.15) and the fact that $1/2 \leq \xi^2 + \zeta^2 \leq 1$ we obtain the global estimate

$$\begin{aligned} & \int_0^T \int_0^1 \left(s\Theta(t)x^{2\alpha-\beta} y_x^2 + s^3\Theta^3(t)x^{2+2\alpha-3\beta} y^2 \right) e^{2s\varphi(t,x)} dx dt \\ & \leq C \left(\int_0^T \int_0^1 f^2(t, x) e^{2s\Phi(t,x)} dx dt + \int_0^T \int_{\omega'} s^3\phi^3 y^2 e^{2s\Phi(t,x)} dx dt \right) \end{aligned} \quad (3.30)$$

This ends the proof of Theorem 3.3. \square

The estimate in Theorem 3.3 was obtained for regular initial data. By density we deduce the following result for the general case: $y_0 \in L^2(0, 1)$.

Corollary 3.6. *Let $T > 0$ be given. Let $\beta \in [\alpha, 1)$ and $\mu \geq \max(0, 2 + 2\alpha - 3\beta)$. Then there exist two positive constants C and s_0 such that every solution y of (1.5)-(1.7) satisfies for all $s \geq s_0$*

$$\begin{aligned} & \int_0^T \int_0^1 (s\Theta x^\alpha y_x^2 + s^3\Theta^3 x^\mu y^2) e^{2s\varphi(t,x)} dx dt \\ & \leq C \left(\int_0^T \int_0^1 f^2(t, x) e^{2s\Phi(t,x)} dx dt + \int_0^T \int_{\omega'} s^3\phi^3 y^2 e^{2s\Phi(t,x)} dx dt \right) \end{aligned} \quad (3.31)$$

Proof. Let $y_0 \in L^2(0, 1)$. By the density of $H_\alpha^1(0, 1)$ in $L^2(0, 1)$, there exist a set $(y_0^n)_n$ in $H_\alpha^1(0, 1)$ which converges to y_0 . Let y^n the unique solution in the space $Z_T := C([0, T], L^2(0, 1)) \cap L^2(0, T; H_\alpha^1)$ of the problem (1.5)-(1.7) associated to the initial data y_0^n . As in (2.13) one has for a constant $C_T > 0$

$$\|(y^m - y^n)(t)\|_{Z_T} := \sup_{[0, T]} \|(y^m - y^n)(t)\|_{L^2}^2 + \int_0^T \|x^{\frac{\alpha}{2}}(y^m - y^n)_x\|_{L^2}^2 dt \leq C_T \|y_0^m - y_0^n\|_{L^2}^2.$$

Therefore the set $(y^n)_n$ has a limit y in the Banach space Z_T . Using classical argument in semigroup theory it is easy to show that y is the solution of the problem (1.5)-(1.7) associated to the initial data

y_0 . On the other hand since $x^\alpha \leq x^{2\alpha-\beta}$ and $x^\mu \leq x^{2+2\alpha-3\beta}$ on $(0,1)$ then we deduce from Theorem 3.3 the estimate

$$\begin{aligned} & \int_0^T \int_0^1 (s\Theta x^\alpha |y_x^n|^2 + s^3\Theta^3 x^\mu |y^n|^2) e^{2s\varphi(t,x)} dx dt \\ & \leq C \left(\int_0^T \int_0^1 f^2(t,x) e^{2s\Phi(t,x)} dx dt + \int_0^T \int_{\omega'} s^3 \phi^3 |y^n|^2 e^{2s\Phi(t,x)} dx dt \right) \end{aligned}$$

And since $s\Theta e^{2s\varphi}$, $s^3\Theta^3 e^{2s\varphi} x^\mu$ and $s^3\phi^3 e^{2s\Phi}$ are bounded then one can pass to the limit and get the desired estimate. \square

For the coupled system (1.9)-(1.12) we prove first an intermediate important result which could be used to show the null controllability for a coupled system with two control forces

Theorem 3.7. *Let $T > 0$ and $(\alpha_1, \alpha_2) \in (0, 1) \times (0, 1)$ be given and suppose that $y_0 \in H_\alpha^1$. Then for all $\beta \in [\max(\alpha_1, \alpha_2), 1[$ there exist two positive constants C and s_0 such that every solution (U, V) of (1.9)-(1.12) satisfies*

$$\begin{aligned} & \int_0^T \int_0^1 s\Theta(t) \left[x^{2\alpha_1-\beta} U_x^2(t, x) + x^{2\alpha_2-\beta} V_x^2(t, x) \right] e^{2s\varphi(t,x)} dx dt \\ & + \int_0^T \int_0^1 s^3\Theta^3(t) \left[x^{2+2\alpha_1-3\beta} U^2(t, x) + x^{2+2\alpha_2-3\beta} V^2(t, x) \right] e^{2s\varphi(t,x)} dx dt \\ & \leq C \int_0^T \int_{\omega'} s^3\Theta^3 \left[U^2(t, x) + V^2(t, x) \right] e^{2s\Phi(t,x)} dx dt \quad \text{for all } s \geq s_0. \end{aligned} \quad (3.32)$$

Proof. Since U is solution of the problem

$$\begin{aligned} U_t - (x^{\alpha_1} U_x)_x &= -b_{11}(t, x)U - b_{21}(t, x)V, & (t, x) \in (0, T) \times (0, 1), \\ U(t, 1) &= U(t, 0) = 0, & t \in (0, T), \\ U(0, x) &= U_0(x), & x \in (0, 1), \end{aligned}$$

then applying the estimate (3.26) to this system we obtain

$$\begin{aligned} & \int_0^T \int_0^1 [s\Theta(t) x^{2\alpha_1-\beta} \xi^2 U_x^2(t, x) + s^3\Theta^3(t) x^{2+2\alpha_1-3\beta} \xi^2 U^2(t, x)] e^{2s\varphi} dx dt \\ & \leq \overline{C} \int_0^T \int_0^1 \xi^2 (b_{11}^2 U^2 + b_{21}^2 V^2) e^{2s\varphi} dx dt + C \int_0^T \int_{\omega'} s\Theta U^2 e^{2s\varphi} dx dt. \end{aligned} \quad (3.33)$$

Using the Hardy-Poincaré inequality (6.56) one has for s large enough

$$\begin{aligned} \int_0^T \int_0^1 b_{11}^2 \xi^2 U^2 e^{2s\varphi} dx dt & \leq C \int_0^T \int_0^1 \left[x^{\alpha_1-\frac{\beta}{2}-1} \xi U e^{s\varphi} \right] \left[x^{1-\alpha_1+\frac{\beta}{2}} \xi U e^{s\varphi} \right] dx dt \\ & \leq C \int_0^T \int_0^1 \left(x^{2\alpha_1-\beta-2} \xi^2 U^2 + x^{2-2\alpha_1+\beta} \xi^2 U^2 \right) e^{2s\varphi} dx dt \\ & \leq C \int_0^T \int_0^1 x^{2\alpha_1-\beta} (\xi U e^{s\varphi})_x^2 dx dt + C \int_0^T \int_0^1 x^{2-2\alpha_1+\beta} \xi^2 U^2 e^{2s\varphi} dx dt \\ & \leq C \int_0^T \int_0^1 \left(x^{2\alpha_1-\beta} \xi^2 U_x^2 + x^{2\alpha_1-\beta} \xi_x^2 U^2 + s^2 \Theta^2 x^{2+2\alpha_1-3\beta} \xi^2 U^2 \right) e^{2s\varphi} dx dt \\ & \quad + C \int_0^T \int_0^1 x^{2-2\alpha_1+\beta} \xi^2 U^2 e^{2s\varphi} dx dt. \end{aligned}$$

So since $\beta \geq \alpha$, ξ_x is supported in ω' and Θ is bounded below then for s large enough we have

$$\begin{aligned} \bar{C} \int_0^T \int_0^1 b_{11}^2 \xi^2 U^2 e^{2s\varphi} dx dt &\leq \frac{1}{4} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_1-\beta} \xi^2 U_x^2 + s^3 \Theta^3(t)x^{2+2\alpha_1-3\beta} \xi^2 U^2] e^{2s\varphi} dx dt \\ &\quad + C \int_0^T \int_{\omega'} U^2 e^{2s\varphi} dx dt \end{aligned} \quad (3.34)$$

Similarly, for s large enough we have

$$\begin{aligned} \bar{C} \int_0^T \int_0^1 b_{21}^2 \xi^2 V^2 e^{2s\varphi} dx dt &\leq \frac{1}{4} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_2-\beta} \xi^2 V_x^2 + s^3 \Theta^3(t)x^{2+2\alpha_2-3\beta} \xi^2 V^2] e^{2s\varphi} dx dt \\ &\quad + C \int_0^T \int_{\omega'} V^2 e^{2s\varphi} dx dt \end{aligned} \quad (3.35)$$

Combining (3.33), (3.34) and (3.35) we deduce the estimate

$$\begin{aligned} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_1-\beta} \xi^2 U_x^2(t, x) + s^3 \Theta^3(t)x^{2+2\alpha_1-3\beta} \xi^2 U^2(t, x)] e^{2s\varphi} dx dt \\ \leq \frac{1}{4} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_1-\beta} \xi^2 U_x^2 + s^3 \Theta^3(t)x^{2+2\alpha_1-3\beta} \xi^2 U^2] e^{2s\varphi} dx dt \\ + \frac{1}{4} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_2-\beta} \xi^2 V_x^2 + s^3 \Theta^3(t)x^{2+2\alpha_2-3\beta} \xi^2 V^2] e^{2s\varphi} dx dt \\ + C \int_0^T \int_{\omega'} s\Theta(U^2 + V^2) e^{2s\varphi} dx dt. \end{aligned} \quad (3.36)$$

For the second component, Arguing as before we have for s large enough

$$\begin{aligned} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_2-\beta} \xi^2 V_x^2(t, x) + s^3 \Theta^3(t)x^{2+2\alpha_2-3\beta} \xi^2 V^2(t, x)] e^{2s\varphi} dx dt \\ \leq \frac{1}{4} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_2-\beta} \xi^2 V_x^2 + s^3 \Theta^3(t)x^{2+2\alpha_2-3\beta} \xi^2 V^2] e^{2s\varphi} dx dt \\ + \frac{1}{4} \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_1-\beta} \xi^2 U_x^2 + s^3 \Theta^3(t)x^{2+2\alpha_1-3\beta} \xi^2 U^2] e^{2s\varphi} dx dt \\ + C \int_0^T \int_{\omega'} s\Theta(U^2 + V^2) e^{2s\varphi} dx dt. \end{aligned} \quad (3.37)$$

Therefore, from (3.36) and (3.37) we deduce the estimate

$$\begin{aligned} &\int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_1-\beta} \xi^2 U_x^2(t, x) + s^3 \Theta^3(t)x^{2+2\alpha_1-3\beta} \xi^2 U^2(t, x)] e^{2s\varphi} dx dt \\ &+ \int_0^T \int_0^1 [s\Theta(t)x^{2\alpha_2-\beta} \xi^2 V_x^2(t, x) + s^3 \Theta^3(t)x^{2+2\alpha_2-3\beta} \xi^2 V^2(t, x)] e^{2s\varphi} dx dt \\ &\leq C \int_0^T \int_{\omega'} s\Theta(U^2 + V^2) e^{2s\varphi} dx dt. \end{aligned} \quad (3.38)$$

This gives an estimate on $(0, a')$. As above, to obtain an estimate on $(a', 1)$, we apply (3.28) to each equation of the system (1.9)-(1.12), we use Hardy-Poincaré inequality and we obtain the estimate

$$\begin{aligned} & \int_0^T \int_0^1 [s\phi\zeta^2(U_x^2 + V_x^2) + s^3\phi^3\zeta^2(U^2 + V^2)] e^{2s\Phi} dx dt \\ & \leq C \int_0^T \int_{\omega'} s^3\phi^3(U^2 + V^2) e^{2s\Phi} dx dt. \end{aligned} \quad (3.39)$$

Consequently, using (3.38), (3.39) and (3.29) we deduce the global estimate

$$\begin{aligned} & \int_0^T \int_0^1 s\Theta(t) [x^{2\alpha_1-\beta}U_x^2(t, x) + x^{2\alpha_2-\beta}V_x^2(t, x)] e^{2s\varphi} dx dt \\ & + \int_0^T \int_0^1 s^3\Theta^3(t) [x^{2+2\alpha_1-3\beta}U^2(t, x) + x^{2+2\alpha_2-3\beta}V^2(t, x)] e^{2s\varphi} dx dt \\ & \leq C \int_0^T \int_{\omega'} s^3\Theta^3 [U^2(t, x) + V^2(t, x)] e^{2s\Phi} dx dt. \end{aligned}$$

This ends the proof. \square

As above, using density argument we deduce the following result for the general case: $U_0, V_0 \in L^2(0, 1)$.

Corollary 3.8. *Let $T > 0$ and $(\alpha_1, \alpha_2) \in (0, 1) \times (0, 1)$ be given. Let $\beta \in [\max(\alpha_1, \alpha_2), 1[$ and $\mu_i \geq \max(0, 2 + 2\alpha_i - 3\beta)$. Then, there exist two positive constants C and s_0 such that every solution (U, V) of (1.9)-(1.12) satisfies*

$$\begin{aligned} & \int_0^T \int_0^1 s\Theta(t) [x^{\alpha_1}U_x^2(t, x) + x^{\alpha_2}V_x^2(t, x)] e^{2s\varphi(t, x)} dx dt \\ & + \int_0^T \int_0^1 s^3\Theta^3(t) [x^{\mu_1}U^2(t, x) + x^{\mu_2}V^2(t, x)] e^{2s\varphi(t, x)} dx dt \\ & \leq C \int_0^T \int_{\omega'} s^3\Theta^3(t) [U^2(t, x) + V^2(t, x)] e^{2s\Phi(t, x)} dx dt \quad \text{for all } s \geq s_0. \end{aligned} \quad (3.40)$$

To study of the null-controllability of the system (1.1)-(1.4) we need to show the following Carleman estimate.

Theorem 3.9. *Let $T > 0$ be given. Assume moreover that*

$$b_{21} \geq \mu \text{ on } [0, T] \times \omega_1 \quad \text{for some } \omega_1 \Subset \omega \text{ and } \mu > 0. \quad (3.41)$$

Then there exist two positive constants C and s_0 such that, every solution (U, V) of (1.9)-(1.12) satisfies for all $s \geq s_0$ the estimates

$$\begin{aligned} & \int_0^T \int_0^1 s\Theta(t) [x^{2\alpha_1-\beta}U_x^2(t, x) + x^{2\alpha_2-\beta}V_x^2(t, x)] e^{2s\varphi(t, x)} dx dt \\ & + \int_0^T \int_0^1 s^3\Theta^3(t) [x^{2+2\alpha_1-3\beta}U^2(t, x) + x^{2+2\alpha_2-3\beta}V^2(t, x)] e^{2s\varphi(t, x)} dx dt \\ & \leq C \int_0^T \int_{\omega} U^2(t, x) dx dt. \end{aligned} \quad (3.42)$$

Remark 3.10. *The assumption (3.41) can be replaced by*

$$b_{21} \leq -\mu \text{ on } [0, T] \times \omega_1 \quad \text{for some } \omega_1 \Subset \omega \text{ and } \mu > 0.$$

Theorem 3.9 is a consequence of Theorem 3.7 applied to ω_1 and the following lemma, see also the proofs of ([8], Theorem 3.2), [29] and [1].

Lemma 3.11. *Suppose moreover that (3.41) holds. Then for all $\varepsilon > 0$ there exists a positive constant $C_\varepsilon > 0$ such that every solution (U, V) of (1.9)-(1.12) satisfies*

$$\int_0^T \int_{\omega_1} s^3 \Theta^3 V^2 e^{2s\Phi} dx dt \leq \varepsilon J(V) + C_\varepsilon \int_0^T \int_{\omega} U^2 dx dt, \quad (3.43)$$

where ω_1 is defined in (3.41) and

$$J(V) := \int_0^T \int_0^1 \left[s \Theta(t) x^{2\alpha_2 - \beta} V_x^2 + s^3 \Theta^3(t) x^{2+2\alpha_2 - 3\beta} V^2 \right] e^{2s\varphi(t,x)} dx dt$$

Proof. Let $\chi \in \mathcal{C}^\infty(0, 1)$ such that $\text{supp } \chi \subset \omega$ and $\chi \equiv 1$ on ω_1 . Multiplying the equation (1.9) by $s^3 \Theta^3 \chi e^{2s\Phi} V$ and integrating, we obtain

$$\begin{aligned} \int_0^T \int_0^1 \chi b_{21} s^3 \Theta^3 e^{2s\Phi} V^2 dx dt &= - \int_0^T \int_0^1 \chi s^3 \Theta^3 e^{2s\Phi} V U_t dx dt + \int_0^T \int_0^1 \chi s^3 \Theta^3 e^{2s\Phi} V (x^{\alpha_1} U_x)_x dx dt \\ &\quad - \int_0^T \int_0^1 \chi b_{11} s^3 \Theta^3 e^{2s\Phi} UV dx dt \end{aligned} \quad (3.44)$$

Integrating by parts and using the equation (1.10), we obtain

$$\begin{aligned} \int_0^T \int_0^1 \chi s^3 \Theta^3 e^{2s\Phi} V U_t dx dt &= \int_0^T \int_0^1 \chi x^{\alpha_2} s^3 \Theta^3 e^{2s\Phi} U_x V_x dx dt + \int_0^T \int_0^1 x^{\alpha_2} s^3 \Theta^3 (\chi e^{2s\Phi})_x UV_x dx dt \\ &\quad + \int_0^T \int_0^1 \chi b_{12} s^3 \Theta^3 e^{2s\Phi} U^2 dx dt + \int_0^T \int_0^1 \chi b_{22} s^3 \Theta^3 e^{2s\Phi} UV dx dt \\ &\quad - \int_0^T \int_0^1 \chi s^3 (\Theta^3 e^{2s\Phi})_t UV dx dt, \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} \int_0^T \int_0^1 \chi s^3 \Theta^3 e^{2s\Phi} V (x^{\alpha_1} U_x)_x dx dt &= - \int_0^T \int_0^1 x^{\alpha_1} \chi s^3 \Theta^3 e^{2s\Phi} U_x V_x dx dt \\ &\quad + \int_0^T \int_0^1 s^3 \Theta^3 x^{\alpha_1} (\chi e^{2s\Phi})_x UV_x dx dt + \int_0^T \int_0^1 s^3 \Theta^3 (x^{\alpha_1} (\chi e^{2s\Phi})_x)_x UV dx dt. \end{aligned} \quad (3.46)$$

So combining the identities (3.44)-(3.46), we get

$$\begin{aligned} \int_0^T \int_0^1 b_{21} \chi s^3 \Theta^3 e^{2s\Phi} V^2 dx dt &= - \underbrace{\int_0^T \int_0^1 (x^{\alpha_1} + x^{\alpha_2}) \chi s^3 \Theta^3 e^{2s\Phi} U_x V_x dx dt}_{I_1} \\ &\quad + \underbrace{\int_0^T \int_0^1 (x^{\alpha_1} - x^{\alpha_2}) s^3 \Theta^3 (\chi e^{2s\Phi})_x UV_x dx dt}_{I_2} - \underbrace{\int_0^T \int_0^1 b_{12} \chi s^3 \Theta^3 e^{2s\Phi} U^2 dx dt}_{I_3} \\ &\quad + \underbrace{\int_0^T \int_0^1 [s^3 \chi (\Theta^3 e^{2s\Phi})_t + s^3 \Theta^3 (x^{\alpha_1} (\chi e^{2s\Phi})_x)_x - (b_{11} + b_{22}) \chi s^3 \Theta^3 e^{2s\Phi}] UV dx dt}_{I_4}. \end{aligned} \quad (3.47)$$

Now we estimate the integrals I_1 , I_2 , I_3 and I_4 . We have

$$\begin{aligned} \left| \int_0^T \int_0^1 x^{\alpha_i} \chi s^3 \Theta^3 e^{2s\Phi} U_x V_x dx dt \right| &= \left| \int_0^T \int_0^1 \left[s^{\frac{1}{2}} \Theta^{\frac{1}{2}} x^{\alpha_2 - \frac{\beta}{2}} e^{s\varphi} V_x \right] \left[s^{\frac{5}{2}} \Theta^{\frac{5}{2}} \chi x^{\alpha_i - \alpha_2 + \frac{\beta}{2}} e^{s(2\Phi - \varphi)} U_x \right] dx dt \right| \\ &\leq \varepsilon \int_0^T \int_0^1 s \Theta x^{2\alpha_2 - \beta} e^{2s\varphi} V_x^2 dx dt + \underbrace{\frac{1}{4\varepsilon} \int_0^T \int_0^1 s^5 \Theta^5 \chi^2 x^{2\alpha_i - 2\alpha_2 + \beta} e^{2s(2\Phi - \varphi)} U_x^2 dx dt}_K. \end{aligned} \quad (3.48)$$

The last integral K should be estimated by an integral in U^2 . For this, we multiply the equation (1.9) by $s^5 \Theta^5 \chi^2 x^\mu e^{2s(2\Phi - \varphi)} U$ where $\mu := 2\alpha_i - \alpha_1 - 2\alpha_2 + \beta$, we integrate by parts and we obtain

$$\begin{aligned} K &= \underbrace{\frac{1}{2} \int_0^T \int_0^1 s^5 \left(\Theta^5 e^{2s(2\Phi - \varphi)} \right)_t \chi^2 x^\mu U^2 dx dt}_{K_1} \\ &\quad + \underbrace{\frac{1}{2} \int_0^T \int_0^1 s^5 \Theta^5 (x^{\alpha_1} (\chi^2 x^\mu e^{2s(2\Phi - \varphi)})_x)_x U^2 dx dt}_{K_2} - \underbrace{\int_0^T \int_0^1 b_{11} s^5 \Theta^5 \chi^2 x^\mu e^{2s(2\Phi - \varphi)} U^2 dx dt}_{K_3} \\ &\quad - \underbrace{\int_0^T \int_0^1 b_{21} s^5 \Theta^5 \chi^2 x^\mu e^{2s(2\Phi - \varphi)} UV dx dt}_{K_4}. \end{aligned}$$

Since $|\Theta'| \leq C\Theta^2$ and $\text{supp}\chi \subset \omega$ we have for $i \in \{1, 2, 3\}$

$$|K_i| \leq C \int_0^T \int_\omega s^7 \Theta^7 e^{2s(2\Phi - \varphi)} U^2 dx dt,$$

For $i = 4$ we have

$$\begin{aligned} |K_4| &= \int_0^T \int_0^1 [s^{\frac{3}{2}} \Theta^{\frac{3}{2}} x^{1+\alpha_2 - \frac{3\beta}{2}} e^{s\varphi} V] [s^{\frac{7}{2}} \Theta^{\frac{7}{2}} b_{21} \chi^2 x^{\mu-1-\alpha_2 + \frac{3\beta}{2}} e^{s(4\Phi - 3\varphi)} U] dx dt \\ &\leq \varepsilon^2 \int_0^T \int_0^1 s^3 \Theta^3 x^{2+2\alpha_2 - 3\beta} e^{2s\varphi} V^2 dx dt + C_\varepsilon \int_0^T \int_\omega s^7 \Theta^7 e^{2s(4\Phi - 3\varphi)} U^2 dx dt. \end{aligned}$$

So, thanks to (3.14) we have

$$|K| \leq \varepsilon^2 \int_0^T \int_0^1 s^3 \Theta^3 x^{2+2\alpha_2 - 3\beta} e^{2s\varphi} V^2 dx dt + C_\varepsilon \int_0^T \int_\omega U^2 dx dt. \quad (3.49)$$

From (3.48)-(3.49) we deduce the estimate

$$|I_1| \leq 2\varepsilon \int_0^T \int_0^1 s \Theta x^{2\alpha_2 - \beta} e^{2s\varphi} V_x^2 dx dt + \frac{\varepsilon}{2} \int_0^T \int_0^1 s^3 \Theta^3 x^{2+2\alpha_2 - 3\beta} e^{2s\varphi} V^2 dx dt + C_\varepsilon \int_0^T \int_\omega U^2 dx dt. \quad (3.50)$$

Similarly we have

$$\begin{aligned} |I_2| &\leq C \int_0^T \int_{\omega} s^4 \Theta^4 |UV_x| e^{2s\Phi} dx dt \\ &\leq \varepsilon \int_0^T \int_0^1 s \Theta x^{2\alpha_2 - \beta} e^{2s\varphi} V_x^2 dx dt + C_{\varepsilon} \int_0^T \int_{\omega} U^2 dx dt, \end{aligned} \quad (3.51)$$

$$|I_3| \leq C \int_0^T \int_{\omega} U^2 dx dt, \quad (3.52)$$

$$\begin{aligned} |I_4| &\leq C \int_0^T \int_{\omega} s^6 \Theta^6 |UV| e^{2s\Phi} dx dt \\ &\leq \varepsilon \int_0^T \int_0^1 s^3 \Theta^3 x^{2+2\alpha_2-3\beta} e^{2s\varphi} V^2 dx dt + C_{\varepsilon} \int_0^T \int_{\omega} U^2 dx dt. \end{aligned} \quad (3.53)$$

Consequently, from the estimates (3.50)-(3.53), we conclude that

$$\int_0^T \int_0^1 b_{21} \chi e^{2s\varphi} V^2 dx dt \leq 3\varepsilon J(V) + C_{\varepsilon} \int_0^T \int_{\omega} U^2 dx dt.$$

Finally, since $\chi \equiv 1$ on ω_1 , then using (3.41) we achieve the claim. \square

As above, using a density argument we deduce the following result for the general case: $U_0, V_0 \in L^2(0, 1)$.

Corollary 3.12. *Let $T > 0$ be given. Assume moreover that (3.41) holds. Let $(\alpha_1, \alpha_2) \in (0, 1) \times (0, 1)$, $\beta \in [\max(\alpha_1, \alpha_2), 1[$ and $\mu_i \geq \max(0, 2 + 2\alpha_i - 3\beta)$. Then, there exist two positive constants C and s_0 such that, every solution (U, V) of (1.9)-(1.12) satisfies, for all $s \geq s_0$ the estimates*

$$\begin{aligned} &\int_0^T \int_0^1 s \Theta(t) [x^{\alpha_1} U_x^2(t, x) + x^{\alpha_2} V_x^2(t, x)] e^{2s\varphi(t, x)} dx dt \\ &+ \int_0^T \int_0^1 s^3 \Theta^3(t) [x^{\mu_1} U^2(t, x) + x^{\mu_2} V^2(t, x)] e^{2s\varphi(t, x)} dx dt \\ &\leq C \int_0^T \int_{\omega} U^2(t, x) dx dt. \end{aligned} \quad (3.54)$$

4. Observability and null controllability of linear systems

As a consequence of the Carleman estimates established in the above section, we prove first a observability inequality for the adjoint problem (1.9)-(1.12) of problem (1.1)-(1.4).

Theorem 4.1. *Let $T > 0$ be given. Assume that (3.41) is satisfied. Then, there exists a positive constant C such that every solution (U, V) of (1.9)-(1.12) satisfies*

$$\int_0^1 [U^2(T, x) + V^2(T, x)] dx \leq C \int_0^T \int_{\omega} U^2(t, x) dx dt. \quad (4.55)$$

Proof. Multiplying the equations (1.9) and (1.10) respectively by U_t and V_t and integrating over $(0, 1)$ the sum of the new equations we obtain

$$\begin{aligned} 0 &= \int_0^1 [U_t^2 + V_t^2] dx - [x^{\alpha_1} U_x U_t]_{x=0}^1 - [x^{\alpha_2} V_x V_t]_{x=0}^1 \\ &\quad + \int_0^1 b_{11} U U_t dx + \int_0^1 b_{22} V V_t + \int_0^1 b_{21} V U_t dx \\ &\quad + \int_0^1 b_{12} U V_t dx + \frac{1}{2} \frac{d}{dt} \int_0^1 [x^{\alpha_1} U_x^2 + x^{\alpha_2} V_x^2] dx. \end{aligned}$$

Using the Young's inequality we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 [x^{\alpha_1} U_x^2 + x^{\alpha_2} V_x^2] dx &\leq \int_0^1 (b_{11}^2 + b_{12}^2) U^2 dx + \int_0^1 (b_{22}^2 + b_{21}^2) V^2 dx \\ &\leq C \int_0^1 (U^2(t, x) + V^2(t, x)) dx \\ &\leq C \int_0^1 [x^{\alpha_1-2} U^2(t, x) + x^{\alpha_2-2} V^2(t, x)] dx. \end{aligned}$$

Hence, using the Hardy-Poincaré inequality (6.56) one has

$$\frac{d}{dt} \int_0^1 [x^{\alpha_1} U_x^2 + x^{\alpha_2} V_x^2] dx \leq C_0 \int_0^1 [x^{\alpha_1} U_x^2 + x^{\alpha_2} V_x^2] dx$$

Hence

$$\frac{d}{dt} \left\{ e^{-C_0 t} \int_0^1 [x^{\alpha_1} U_x^2 + x^{\alpha_2} V_x^2] dx \right\} \leq 0.$$

Consequently, the function $t \mapsto e^{-C_0 t} \int_0^1 [x^{\alpha_1} U_x^2 + x^{\alpha_2} V_x^2] dx$ is not increasing. Thus,

$$\int_0^1 [x^{\alpha_1} U_x^2(T, x) + x^{\alpha_2} V_x^2(T, x)] dx \leq e^{C_0 T} \int_0^1 [x^{\alpha_1} U_x^2(t, x) + x^{\alpha_2} V_x^2(t, x)] dx.$$

Integrating over $[\frac{T}{4}, \frac{3T}{4}]$ and using the Carleman estimate (3.54) one obtains

$$\begin{aligned} \int_0^1 [x^{\alpha_1} U_x^2(T, x) + x^{\alpha_2} V_x^2(T, x)] dx &\leq \frac{2e^{C_0 T}}{T} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 [x^{\alpha_1} U_x^2(t, x) + x^{\alpha_2} V_x^2(t, x)] dx dt \\ &\leq C_T \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 s \Theta e^{2s\varphi} [x^{\alpha_1} U_x^2(t, x) + x^{\alpha_2} V_x^2(t, x)] dx dt \\ &\leq C_T \int_0^T \int_{\omega} U^2(t, x) dx dt, \end{aligned}$$

On the other hand, using hardy-Poincaré inequality one gets

$$\begin{aligned} \int_0^1 [U^2(T, x) + V^2(T, x)] dx &\leq \int_0^1 [x^{\alpha_1-2} U^2(T, x) + x^{\alpha_2-2} V^2(T, x)] dx \\ &\leq C \int_0^1 [x^{\alpha_1} U_x^2(T, x) + x^{\alpha_2} V_x^2(T, x)] dx \end{aligned}$$

This ends the proof. \square

By Theorem 4.1 and a classical argument one can deduce the controllability result

Theorem 4.2. *If the assumption (3.41) is satisfied, then the degenerate coupled system (1.1)-(1.4) is null controllable.*

5. Appendix 1

As in [2], [8], [1], we give the proof of the Caccioppoli's inequality for degenerate coupled systems with two different diffusion coefficients.

Lemma 5.1. *Let $\omega' \Subset \omega$. Then there exists a positive constant C such that*

$$\int_0^T \int_{\omega'} [U_x^2(t, x) + V_x^2(t, x)] e^{2s\varphi_i} dx dt \leq C \int_0^T \int_{\omega} [U^2(t, x) + V^2(t, x)] e^{2s\varphi_i} dx dt.$$

Proof. Let $\chi \in C^\infty(0, 1)$ such that $\text{supp } \chi \subset \omega$ and $\chi \equiv 1$ on ω' . We have

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \left[\int_0^1 \chi^2 (U^2 + V^2) e^{2s\varphi_i} dx \right] dt \\ &= -2 \int_0^T \int_0^1 \chi^2 x^{\alpha_1} U_x^2 e^{2s\varphi_i} dx dt - 2 \int_0^T \int_0^1 \chi^2 x^{\alpha_2} V_x^2 e^{2s\varphi_i} dx \\ &\quad + \int_0^T \int_0^1 (x^{\alpha_1} (\chi^2 e^{2s\varphi_i})_x)_x U^2 dx + \int_0^T \int_0^1 (x^{\alpha_2} (\chi^2 e^{2s\varphi_i})_x)_x V^2 dx \\ &\quad - 2 \int_0^T \int_0^1 b_{11} \chi^2 U^2 e^{2s\varphi_i} dx dt - 2 \int_0^T \int_0^1 b_{22} \chi^2 V^2 e^{2s\varphi_i} dx dt \\ &\quad 2 \int_0^T \int_0^1 s \dot{\varphi}_i \chi^2 (U^2 + V^2) e^{2s\varphi_i} dx dt - 2 \int_0^T \int_0^1 (b_{12} + b_{21}) \chi^2 UV e^{2s\varphi_i} dx dt \end{aligned}$$

Therefore, since χ is supported in ω and $\chi \equiv 1$ in ω' then, using Young inequality one obtains

$$\begin{aligned} \int_0^T \int_{\omega'} (U_x^2 + V_x^2) e^{2s\varphi_i} dx dt &\leq C \int_0^T \int_0^1 \chi^2 (x^{\alpha_1} U_x^2 + x^{\alpha_2} V_x^2) e^{2s\varphi_i} dx dt \\ &\leq C \int_0^T \int_{\omega} (U^2 + V^2) e^{2s\varphi_i} dx dt. \end{aligned}$$

This ends the proof. □

6. Appendix 2

Lemma 6.1. *For all $\gamma < 1$ and all v locally absolutely continuous on $(0, 1]$, continuous at 0 and satisfying $v(0) = 0$ and $\int_0^1 x^\gamma v_x^2 dx < +\infty$ the following Hardy-Poincaré inequality holds*

$$\int_0^1 x^{\gamma-2} v^2 dx \leq C_\gamma \int_0^1 x^\gamma v_x^2 dx \quad \text{where } C_\gamma = \frac{4}{(1-\gamma)^2} \quad (6.56)$$

Proof. This result was proved by Cannarsa et al. in [2] for $\gamma \in (0, 1)$, but by a careful examination of the proof one can see that it remains valid for all $\gamma < 1$. In fact let $\gamma < 1$ and $\delta = \frac{\gamma+1}{2}$. Using Holder

inequality and Fubini's theorem one has

$$\begin{aligned}
\int_0^1 x^{\gamma-2} v^2 dx &= \int_0^1 x^{\gamma-2} \left(\int_0^x y^{\delta/2} v'(y) y^{-\delta/2} dy \right)^2 dx \\
&\leq \int_0^1 x^{\gamma-2} \left(\int_0^x y^{\delta} |v'(y)|^2 dy \right) \left(\int_0^x y^{-\delta} dy \right) dx \\
&= \frac{1}{1-\delta} \int_0^1 \int_0^x x^{\gamma-\delta-1} y^{\delta} |v'(y)|^2 dy dx \\
&= \frac{1}{1-\delta} \int_0^1 \left(\int_y^1 x^{\gamma-\delta-1} dx \right) y^{\delta} |v'(y)|^2 dy \\
&\leq \frac{1}{(1-\delta)(\delta-\gamma)} \int_0^1 y^{\gamma} |v'(y)|^2 dy \\
&= \frac{4}{(1-\gamma)^2} \int_0^1 y^{\gamma} |v'(y)|^2 dy.
\end{aligned}$$

This ends the proof. \square

7. Conclusion

In this paper, we studied the null controllability of linear degenerate systems with two different coefficients diffusion not necessarily of the cascade form. We developed new Carleman estimates. By a standard linearization argument and fixed point, see [1], [2], [9], [35], one can show easily the null controllability of semilinear degenerate coupled systems with two different diffusion coefficients. In this paper we studied coupled system of two weakly degenerate equations. The cases when one of the equation is strongly degenerate systems are open.

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